Solution to Problem Set 1

1. (a) $\mathbb{R} \setminus A = (-\infty, 0) \cup (4, +\infty)$

   (b) $\mathbb{R} \setminus (A \cap B) = (-\infty, 3) \cup (4, +\infty)$

   (c) $\mathbb{R} \setminus \big((A \cup B) \setminus (A \cap B)\big) = (-\infty, 0) \cup [3, 4] \cup (6, +\infty)$

   (d) $A \cup \big(B \cap (\mathbb{R} \setminus A)\big) = [0, 6]$

2. (a) We consider the following two cases:

   ⦿ If $(x+1)(x-2) > 0$, then the given inequality is equivalent to
   $$(x-1)(x+2) \geq 0,$$
   so $x \leq -2$ or $x \geq 1$. Combining this with the assumption $(x+1)(x-2) > 0$, we have
   $$(x \leq -2 \text{ or } x \geq 1) \qquad \text{and} \qquad (x < -1 \text{ or } x > 2).$$
   Therefore in this case we obtain the solutions $x \leq -2$ or $x > 2$, i.e. $x \in (-\infty, -2] \cup (2, +\infty)$.

   ⦿ If $(x+1)(x-2) < 0$, then the given inequality is equivalent to
   $$(x-1)(x+2) \leq 0,$$
   so $-2 \leq x \leq 1$. Combining this with the assumption $(x+1)(x-2) < 0$, we have
   $$(-2 \leq x \leq 1) \qquad \text{and} \qquad (-1 < x < 2).$$
   Therefore in this case we obtain the solutions $-1 < x \leq 1$, i.e. $x \in (-1, 1]$.

   Combining the two cases, the solutions to the inequality are all those $x \in (-\infty, -2] \cup (-1, 1] \cup (2, +\infty)$.

   (b) We consider the following two cases:

   ⦿ If $(x+1)(x-2) > 0$, then the given inequality is equivalent to
   $$x^2 + 1 \geq 0,$$
   which is always true. Now $(x+1)(x-2) > 0$ is equivalent to $(x < -1$ or $x > 2)$, so in this case we obtain the solutions $x \in (-\infty, -1) \cup (2, +\infty)$.

   ⦿ If $(x+1)(x-2) < 0$, then the given inequality is equivalent to
   $$x^2 + 1 \leq 0,$$
   which is always false. Therefore in this case we obtain no solutions to the inequality.

   Combining the two cases, the solutions to the inequality are all those $x \in (-\infty, -1) \cup (2, +\infty)$.

   (c) We consider the following two cases:

   ⦿ If $x > 0$, then the given inequality is equivalent to
   $$x^2 - 3 \geq 2x \qquad \Leftrightarrow \qquad x^2 - 2x - 3 \geq 0 \qquad \Leftrightarrow \qquad (x-3)(x+1) \geq 0,$$
   so $x \leq -1$ or $x \geq 3$. Combining this with $x > 0$, we obtain the solutions $x \geq 3$, i.e. $x \in [3, +\infty)$.

- If $x < 0$, then the given inequality is equivalent to
$$x^2 - 3 \le 2x \quad \Leftrightarrow \quad x^2 - 2x - 3 \le 0 \quad \Leftrightarrow \quad (x-3)(x+1) \le 0,$$
so $-1 \le x \le 3$. Combining this with $x < 0$, we obtain the solutions $-1 \le x < 0$, i.e. $x \in [-1, 0)$.

Combining the two cases, the solutions to the inequality are all those $x \in [-1, 0) \cup [3, +\infty)$.

(d) We consider the following two cases:

- If $(1-x)^{\frac{2}{3}}(2-x)^{\frac{1}{3}} > 0$, then the given inequality is equivalent to
$$4 - x > 0,$$
so $x < 4$. At the same time, the solutions to $(1-x)^{\frac{2}{3}}(2-x)^{\frac{1}{3}} > 0$ are
$$1 - x \ne 0 \text{ and } 2 - x > 0,$$
so $x < 1$ or $1 < x < 2$. Combining these, in this case we obtain the solutions $x < 1$ or $1 < x < 2$, i.e. $x \in (-\infty, 1) \cup (1, 2)$.
- If $(1-x)^{\frac{2}{3}}(2-x)^{\frac{1}{3}} < 0$, then the given inequality is equivalent to
$$4 - x < 0,$$
so $x > 4$. At the same time, the solutions to $(1-x)^{\frac{2}{3}}(2-x)^{\frac{1}{3}} < 0$ are
$$1 - x \ne 0 \text{ and } 2 - x < 0,$$
so $x > 2$. Combining these, in this case we obtain the solutions $x > 4$, i.e. $x \in (4, +\infty)$.

Combining the two cases, the solutions to the inequality are all those $x \in (-\infty, 1) \cup (1, 2) \cup (4, +\infty)$.

(e) We consider the following two cases:

- If $2x - 1 \ge 0$, then the given inequality is equivalent to
$$3 < 2x - 1 \le 5,$$
so $2 < x \le 3$. Combining this with the assumption $2x - 1 \ge 0$, i.e. $x \ge \frac{1}{2}$, in this case we obtain the solutions $2 < x \le 3$, i.e. $x \in (2, 3]$.
- If $2x - 1 < 0$, then the given inequality is equivalent to
$$3 < -(2x - 1) \le 5,$$
so $-2 \le x < -1$. Combining this with the assumption $2x - 1 < 0$, i.e. $x < \frac{1}{2}$, in this case we obtain the solutions $-2 \le x < -1$, i.e. $x \in [-2, -1)$.

Combining the two cases, the solutions to the inequality are all those $x \in [-2, -1) \cup (2, 3]$.

(f) Note that $|x^2 - 4x - 1| > 4$ if and only if either $x^2 - 4x - 1 > 4$ or $x^2 - 4x - 1 < -4$.

- If $x^2 - 4x - 1 > 4$, then we obtain
$$x^2 - 4x - 5 > 0,$$
i.e. $(x-5)(x+1) > 0$. The solutions are ($x < -1$ or $x > 5$), i.e. $x \in (-\infty, -1) \cup (5, +\infty)$.
- If $x^2 - 4x - 1 < -4$, then we obtain
$$x^2 - 4x + 3 < 0,$$
i.e. $(x-3)(x-1) < 0$. The solutions are $1 < x < 3$, i.e. $x \in (1, 3)$.

Combining the two cases, the solutions to the inequality are all those $x \in (-\infty, -1) \cup (1, 3) \cup (5, +\infty)$.

3. (a)

$$\begin{aligned}30+32+34+36+\cdots+50 &= 2(15+16+17+18+\cdots+25)\\ &= 2[(1+2+\cdots+25)-(1+2+\cdots+14)]\\ &= 2\left(\frac{25\times 26}{2}-\frac{14\times 15}{2}\right)=25\times 26-14\times 15=440.\end{aligned}$$

(b)

$$\begin{aligned}&1\cdot 4\cdot 7+2\cdot 5\cdot 8+\cdots+30\cdot 33\cdot 36\\ &=\sum_{k=1}^{30}k(k+3)(k+6)=\sum_{k=1}^{30}(k^3+9k^2+18k)=\sum_{k=1}^{30}k^3+9\sum_{k=1}^{30}k^2+18\sum_{k=1}^{30}k\\ &=\frac{30^2\times 31^2}{4}+9\cdot\frac{30\times 31\times 61}{6}+18\cdot\frac{30\times 31}{2}=309690\end{aligned}$$

(c)

$$1+1.01+1.01^2+1.01^3+\cdots+1.01^{10}=\frac{1.01^{11}-1}{1.01-1}=100(1.01^{11}-1)$$

4. (a)

$$\begin{aligned}\sum_{k=1}^{n}(k+2)(k-3)&=\sum_{k=1}^{n}(k^2-k-6)=\sum_{k=1}^{n}k^2-\sum_{k=1}^{n}k-6\sum_{k=1}^{n}1\\ &=\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}-6n=\frac{n^3-19n}{3}\end{aligned}$$

(b)

$$1+5^{\frac{1}{3}}+5^{\frac{2}{3}}+5+5^{\frac{4}{3}}+\cdots+5^n=\sum_{k=0}^{3n}\left(5^{\frac{1}{3}}\right)^k=\frac{5^{\frac{3n+1}{3}}-1}{5^{\frac{1}{3}}-1}$$

5. (a) Since the volume of the circular cylinder is 100, we have $\pi r^2 h=100$, and so $r^2=\frac{100}{\pi h}$. Therefore the function $r\colon(0,+\infty)\to\mathbb{R}$ is given by

$$r(h)=\sqrt{\frac{100}{\pi h}}=\frac{10}{\sqrt{\pi h}}.$$

(b) Pythagoras' Theorem gives $l^2+l^2+l^2=d^2$, so $l=\sqrt{\frac{d^2}{3}}=\frac{d}{\sqrt{3}}$. (Why not $\pm\sqrt{\frac{d^2}{3}}$? Why $\sqrt{d^2}=d$?)

Now the surface area of the cube $A\colon(0,+\infty)\to\mathbb{R}$ is given by

$$A(d)=6l^2=6\cdot\left(\frac{d^2}{3}\right)=2d^2,$$

and the volume of the cube $V\colon(0,+\infty)\to\mathbb{R}$ is given by

$$V(d)=l^3=\left(\frac{d}{\sqrt{3}}\right)^3=\frac{d^3}{3^{3/2}}.$$

(c) Let the coordinates of P be (x, y). Our goal is to express x and y as functions of m.

Since P lies on the graph of f, we have $y = f(x) = \sqrt{x}$. Thus the slope of the line joining P and $(0,0)$ is

$$m = \frac{y-0}{x-0} = \frac{\sqrt{x}-0}{x-0} = \frac{1}{\sqrt{x}}.$$

Therefore $x, y: (0, +\infty) \to \mathbb{R}$ are given by

$$x(m) = 1/m^2 \qquad \text{and} \qquad y(m) = \sqrt{x(m)} = 1/m.$$

6. (a) The expression $f(x)$ is well-defined if and only if

$$x + 3 \neq 0 \qquad \text{and} \qquad \frac{x+2}{x+3} \geq 0.$$

Both of these hold if and only if $x \geq -2$ or $x < -3$, so the natural domain of f is $(-\infty, -3) \cup [-2, +\infty)$.

(b) The expression $f(x)$ is well-defined if and only if

$$x + 2 \geq 0 \qquad \text{and} \qquad x + 3 > 0.$$

Both of these hold if and only if $x \geq -2$, so the natural domain of f is $[-2, +\infty)$.

(c) The expression $f(x)$ is well-defined if and only if

$$x - 2 \neq 0 \qquad \text{and} \qquad \frac{x^2-1}{x-2} \geq 0.$$

Both of these hold if and only if $\begin{cases} x^2 - 1 \geq 0 \\ x - 2 > 0 \end{cases}$ or $\begin{cases} x^2 - 1 \leq 0 \\ x - 2 < 0 \end{cases}$, i.e. $\begin{cases} x \geq 1 \text{ or } x \leq -1 \\ x > 2 \end{cases}$ or $\begin{cases} -1 \leq x \leq 1 \\ x < 2 \end{cases}$.

This is the same as $x > 2$ or $-1 \leq x \leq 1$, so the natural domain of f is $[-1, 1] \cup (2, +\infty)$.

(d) The expression $f(x)$ is well-defined if and only if $|x| \geq 0$. This always hold for every real number x, so the natural domain of f is $\mathbb{R}$.

(e) The expression $f(x)$ is well-defined if and only if $x \geq 0$. So the natural domain of f is $[0, +\infty)$.

(f) The expression $f(x)$ is well-defined if and only if $x - |x| \neq 0$. This holds if and only if $x < 0$, so the natural domain of f is $(-\infty, 0)$.

(g) The expression $f(x)$ is well-defined if and only if $\sin x \neq 0$. This holds if and only if $x \neq n\pi$ for any integer n. So the natural domain of f can be written as any one of the following:

- $\mathbb{R} \setminus \{n\pi: n \text{ is an integer}\}$, or
- $\cdots \cup (-2\pi, -\pi) \cup (-\pi, 0) \cup (0, \pi) \cup (\pi, 2\pi) \cup (2\pi, 3\pi) \cup \cdots$.

(h) The expression $f(x)$ is well-defined if and only if $|x| > 0$. This always hold for every non-zero real number x, so the natural domain of f is $\mathbb{R} \setminus \{0\}$.

7. (a) The expression $g(x)$ is well-defined if and only if $f(x)$ is well-defined. So the natural domain of g is the same as the domain of f, which is $\mathbb{R}$.
From the given graph we see that the range of f is $[q, +\infty)$. So the range of $g = 1 - f$ is $(-\infty, 1 - q]$.

(b) The expression $h(x)$ is well-defined if and only if $f(x) \geq 0$. From the given graph, we see that $f(x) \geq 0$ if and only if $x \leq a$ or $x \geq b$, so the natural domain of h is $(-\infty, a] \cup [b, +\infty)$.
From the given graph we also see that on the domain $(-\infty, a] \cup [b, +\infty)$, $f(x)$ can take every value in $[0, +\infty)$. So the range of $h = \sqrt[4]{f}$ is $[0, +\infty)$ too.

(c) The expression $m(x)$ is well-defined if and only if $f(x) \neq 0$. From the graph of f, we see that $f(x) \neq 0$ if and only if $x \neq a$ and $x \neq b$, so the natural domain of m is $\mathbb{R} \setminus \{a, b\}$, or $(-\infty, a) \cup (a, b) \cup (b, +\infty)$.
From the graph we also see that on the domain $\mathbb{R} \setminus \{a, b\}$, $f(x)$ takes every value in $[q, 0) \cup (0, +\infty)$. Note that when $f(x) > 0$ we also have $\frac{2}{f(x)} > 0$, while when $q \leq f(x) < 0$ we have $\frac{2}{f(x)} \leq \frac{2}{q}$. So the range of $m = \frac{2}{f}$ is $\left(-\infty, \frac{2}{q}\right] \cup (0, +\infty)$.

8. (a) Let x be a real number.
- ⊙ If $x > a$, then $f(x) = x - a$ and $|x - a| = x - a$, so
$$\frac{1}{2}(x-a) + \frac{1}{2}|x-a| = \frac{1}{2}(x-a) + \frac{1}{2}(x-a) = x - a = f(x).$$
- ⊙ If $x \leq a$, then $f(x) = 0$ and $|x - a| = -(x - a) = a - x$, so
$$\frac{1}{2}(x-a) + \frac{1}{2}|x-a| = \frac{1}{2}(x-a) - \frac{1}{2}(x-a) = 0 = f(x).$$

Therefore $f(x) = \frac{1}{2}(x-a) + \frac{1}{2}|x-a|$ for every real number x. ■

(b) Note that the piecewise defined function P is the sum of the following three functions
$$P_1(x) = 29,$$
$$P_2(x) = \begin{cases} 10.5x + 8 - 29 & \text{if } x > 2 \\ 0 & \text{if } 0 < x \leq 2 \end{cases} = \begin{cases} 10.5(x-2) & \text{if } x > 2 \\ 0 & \text{if } 0 < x \leq 2 \end{cases},$$
$$P_3(x) = \begin{cases} (7x + 39.5) - (10.5x + 8) & \text{if } x > 9 \\ 0 & \text{if } 0 < x \leq 9 \end{cases} = \begin{cases} -3.5(x-9) & \text{if } x > 9 \\ 0 & \text{if } 0 < x \leq 9 \end{cases}.$$
So applying the result from (a) to the functions P_2 and P_3, we obtain
$$\begin{aligned} P(x) &= P_1(x) + P_2(x) + P_3(x) \\ &= 29 + 10.5\left[\frac{1}{2}(x-2) + \frac{1}{2}|x-2|\right] + (-3.5)\left[\frac{1}{2}(x-9) + \frac{1}{2}|x-9|\right] \\ &= (3.5x + 34.25) + 5.25|x-2| - 1.75|x-9|. \end{aligned}$$

9. (a) $f:[1,5]\to\mathbb{R}$ defined by

$$f(x)=\frac{5x-11}{2}.$$

(b) $f:(-\infty,0]\to\mathbb{R}$ defined by $f(x)=1-\sqrt{-x}$.

(c) $f:[-4,4]\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}-\dfrac{3}{2}x-3 & \text{if } x\in[-4,-2]\\ \sqrt{4-x^2} & \text{if } x\in(-2,2)\\ \dfrac{3}{2}x-3 & \text{if } x\in[2,4]\end{cases}.$$

10. (a) For every $x\in\mathbb{R}$, we have

$$f(-x)=(-x)^5+(-x)^3+(-x)=-x^5-x^3-x=-(x^5+x^3+x)=-f(x),$$

so f is odd. Since $f(-1)=-3\neq f(1)$, f is not even.

(b) For every $x\in\mathbb{R}$, we have

$$g(-x)=\sqrt{|-x|}=\sqrt{|x|}=g(x),$$

so g is even. Since $g(-4)=2\neq -g(4)$, g is not odd.

(c) For every rational number x, the number $-x$ is also rational; so

$$h(-x)=1=h(x).$$

For every irrational number x, the number $-x$ is also irrational; so

$$h(-x)=-1=h(x).$$

Therefore $h(-x)=h(x)$ for every $x\in\mathbb{R}$, which implies that h is even. Since $h(-2)=1\neq -h(2)$, h is not odd.

11. Given any function $f:\mathbb{R}\to\mathbb{R}$, we define two functions $f_1,f_2:\mathbb{R}\to\mathbb{R}$ by

$$f_1(x)=\frac{f(x)-f(-x)}{2} \qquad \text{and} \qquad f_2(x)=\frac{f(x)+f(-x)}{2} \qquad \text{for every } x\in\mathbb{R}.$$

Then,

- For every $x\in\mathbb{R}$, we have $f_1(-x)=\frac{f(-x)-f(x)}{2}=-\frac{f(x)-f(-x)}{2}=-f_1(x)$. So f_1 is odd.
- For every $x\in\mathbb{R}$, we have $f_2(-x)=\frac{f(-x)+f(x)}{2}=\frac{f(x)+f(-x)}{2}=f_2(x)$. So f_2 is even.
- For every $x\in\mathbb{R}$, we have $f_1(x)+f_2(x)=\frac{f(x)-f(-x)}{2}+\frac{f(x)+f(-x)}{2}=f(x)$. So $f=f_1+f_2$. ∎

12. (a) For every $x \in \mathbb{R}$, we have

$$\begin{aligned} g(-x) &= |f(-x)| = |-f(x)| && (f \text{ is an odd function}) \\ &= |f(x)| && (\text{property of absolute value}) \\ &= g(x). \end{aligned}$$

So g is even. ∎

(b) For every $x \in \mathbb{R}$, we have

$$\begin{aligned} g(-x) &= f(|-x|) = f(|x|) && (\text{property of absolute value}) \\ &= g(x). \end{aligned}$$

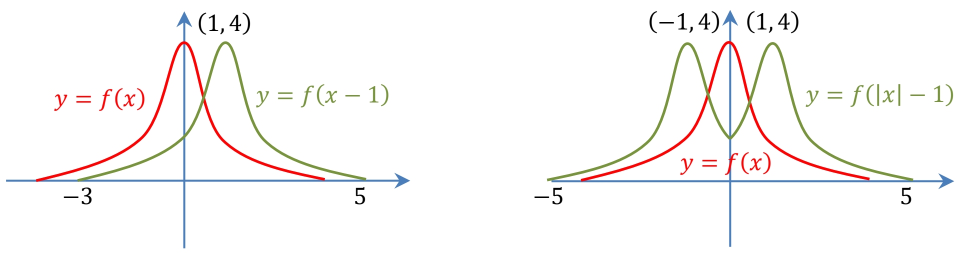
So g is even. ∎

(c) For every $x \in \mathbb{R}$, we have

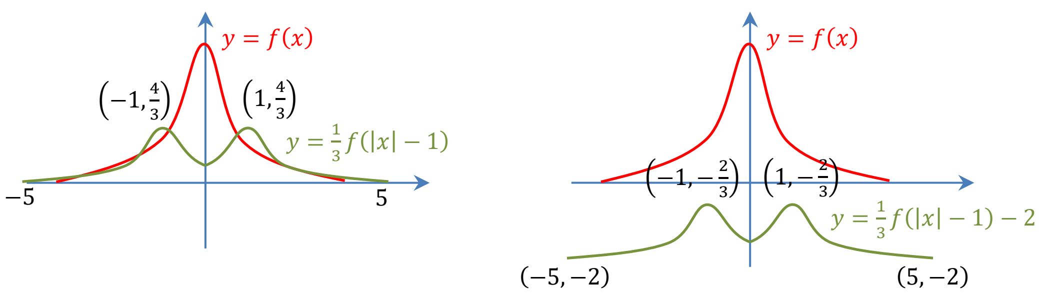
$$\begin{aligned} (f \circ g)(-x) &= f\big(g(-x)\big) = f\big(-g(x)\big) && (g \text{ is an odd function}) \\ &= f\big(g(x)\big) && (f \text{ is an even function}) \\ &= (f \circ g)(x). \end{aligned}$$

So $f \circ g$ is even. ∎

13. We start from the graph of $y = f(x)$. A right translation of 1 unit gives the graph of $y = f(x-1)$, and replacing the part of the graph lying on the left of the y-axis by the reflection of the part on the right across the y-axis, we get the graph of $y = f(|x|-1)$.



Next, a vertical compression by a factor of 3 gives the graph of $y = \frac{1}{3}f(|x|-1)$, and finally a downward translation of 2 units gives the desired graph of $y = \frac{1}{3}f(|x|-1) - 2$.



14. (a) If m and n are positive integers with $f(m) = f(n)$, then

$$\text{the } m^{\text{th}} \text{ prime number} = \text{the } n^{\text{th}} \text{ prime number}.$$

This obviously implies that $m = n$. So f is one-to-one.

(b) For the two distinct real numbers -1 and 1, we have

$$f(-1) = \ln[1 + (-1)^2] = \ln(1 + 1^2) = f(1).$$

So f is not one-to-one.

(c) If a and b are numbers in $(-1, +\infty)$ with $f(a) = f(b)$, then

$$\begin{aligned} & \frac{a}{\sqrt{a+1}} = \frac{b}{\sqrt{b+1}} \\ \Rightarrow \quad & a\sqrt{b+1} = b\sqrt{a+1} \\ \Rightarrow \quad & a^2(b+1) = b^2(a+1) \\ \Rightarrow \quad & a^2b - ab^2 + a^2 - b^2 = 0 \\ \Rightarrow \quad & (ab + a + b)(a - b) = 0. \end{aligned}$$

Now there are two possible cases:

- ⊙ If the second factor $a - b$ is zero, then we have $a = b$.
- ⊙ If the first factor $ab + a + b$ were zero, then we must have $(a+1)(b+1) = 1$. Since the numbers $a + 1$ and $b + 1$ are both positive, it must follow that one of them is ≥ 1 and the other is ≤ 1, which implies that either $a = b = 0$ or a and b are of opposite signs. But on the other hand, $\frac{a}{\sqrt{a+1}} = \frac{b}{\sqrt{b+1}}$ implies that the numbers a and b must have the same sign. Therefore we must have $a = b = 0$.

In any case, we always have $a = b$. Therefore f is one-to-one.

(d) We observe that every horizontal line has either no intersection or just one point of intersection with the graph of f. So by the horizontal line test, the function f is one-to-one.

15. (a) Suppose that $a, b \in \mathbb{R} \setminus \{-2\}$ satisfies $f(a) = f(b)$. Then we have $\frac{2-a}{2+a} = \frac{2-b}{2+b}$, which implies that $(2 - a)(2 + b) = (2 - b)(2 + a)$. Thus $4 - 2a + 2b - ab = 4 - 2b + 2a - ab$, and therefore $4a = 4b$; i.e. $a = b$. This shows that f is one-to-one and so f has an inverse.

To find the inverse of f, we write $y = f(x)$, i.e.

$$y = \frac{2 - x}{2 + x} = -1 + \frac{4}{2 + x}.$$

From this expression we see that $y \neq -1$. Now the above also implies that $2 + x = \frac{4}{1+y}$, and so

$$x = -2 + \frac{4}{1 + y} = \frac{2 - 2y}{1 + y}.$$

Therefore the inverse $f^{-1}: \mathbb{R} \setminus \{-1\} \to \mathbb{R} \setminus \{-2\}$ is given by

$$f^{-1}(x) = \frac{2 - 2x}{1 + x}.$$

(b) Suppose that a and b are real numbers and $f(a) = f(b)$. Then $\frac{e^a}{1+e^a} = \frac{e^b}{1+e^b}$, i.e. $1 - \frac{1}{1+e^a} = 1 - \frac{1}{1+e^b}$. This implies that $\frac{1}{1+e^a} = \frac{1}{1+e^b}$, so $e^a = e^b$. Since the exponential function is one-to-one, we have $a = b$ and so f is one-to-one, i.e. f has an inverse (whose domain is the range of f).

To find the inverse of f, we write $y = f(x)$, i.e. $y = \frac{e^x}{1+e^x} = 1 - \frac{1}{1+e^x}$. Then

$$1 - y = \frac{1}{1+e^x}$$

from which we deduce that $y \in (0, 1)$ because $e^x > 0$. Finally for each $y \in (0, 1)$ we have $1 + e^x = \frac{1}{1-y}$, and so

$$x = \ln\left(\frac{1}{1-y} - 1\right) = \ln\frac{y}{1-y}.$$

Therefore the inverse $f^{-1}: (0, 1) \to \mathbb{R}$ is given by

$$f^{-1}(x) = \ln\frac{x}{1-x}.$$

(c) Suppose that $a, b \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ satisfy $f(a) = f(b)$, i.e. $\tan a = \tan b$. Then $b = n\pi + a$ for some integer n. But since a and b both belong to $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, we must have $b - a \in (-\pi, \pi)$ and so the integer n must be 0. Thus $a = b$ and f is one-to-one, i.e. f has an inverse (whose domain is the range of f).

To find the inverse of f, we write $y = g(x)$, i.e.

$$y = \tan x.$$

The general solution to this equation is

$$x = n\pi + \arctan y.$$

where n is an integer. But $\arctan y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and since we only look for solutions $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, we must have $n = 1$ which gives the solution

$$x = \pi + \arctan y,$$

for each $y \in \mathbb{R}$. So the inverse $f^{-1}: \mathbb{R} \to \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ is given by

$$f^{-1}(x) = \pi + \arctan x.$$

(d) Since -1 and 1 are two distinct numbers in the interval $\left(-\sqrt{\frac{\pi}{2}}, \sqrt{\frac{\pi}{2}}\right)$ and

$$f(-1) = \tan(-1)^2 = \tan 1^2 = f(1),$$

we see that f is not one-to-one, so f does not have an inverse.

16. (a) We solve the given equation.

$$\begin{aligned}\cos 2x + 3\sin x - 2 &= 0\\ (1 - 2\sin^2 x) + 3\sin x - 2 &= 0\\ -2\sin^2 x + 3\sin x - 1 &= 0\\ -(\sin x - 1)(2\sin x - 1) &= 0\end{aligned}$$

This gives $\sin x = 1$ or $\sin x = \frac{1}{2}$. Therefore the general solution is

$$x = 2n\pi + \frac{\pi}{2} \qquad \text{or} \qquad x = n\pi + (-1)^n \frac{\pi}{6}$$

where n is an integer.

(b) We solve the given equation.

$$\begin{aligned}\cos\left(x - \frac{\pi}{3}\right) + 2\cos\left(x + \frac{\pi}{3}\right) &= 0\\ \left(\cos x \cos\frac{\pi}{3} + \sin x \sin\frac{\pi}{3}\right) + 2\left(\cos x \cos\frac{\pi}{3} - \sin x \sin\frac{\pi}{3}\right) &= 0\\ 3\cos x \cos\frac{\pi}{3} &= \sin x \sin\frac{\pi}{3}\\ \tan x = 3\cot\frac{\pi}{3} &= \sqrt{3}\end{aligned}$$

Therefore the solutions in $[0, 4\pi]$ are

$$x = \frac{\pi}{3}, \qquad x = \frac{4\pi}{3}, \qquad x = \frac{7\pi}{3} \qquad \text{or} \qquad x = \frac{10\pi}{3}.$$

17. (a) Whenever each term is well-defined, we have

$$\begin{aligned}\frac{\sec(-x) + \sin\left(-\frac{\pi}{2} - x\right)}{\csc(3\pi - x) - \cos\left(\frac{3\pi}{2} + x\right)} &= \frac{\sec x - \cos x}{\csc x - \sin x} = \frac{\sec x - \cos x}{\csc x - \sin x} \cdot \frac{\cos x}{\sin x} \cdot \frac{\sin x}{\cos x}\\ &= \frac{1 - \cos^2 x}{1 - \sin^2 x} \cdot \frac{\sin x}{\cos x} = \frac{\sin^2 x}{\cos^2 x} \cdot \frac{\sin x}{\cos x} = \tan^3 x.\end{aligned}$$

(b) Whenever each term is well-defined, we have

$$\begin{aligned}\frac{\cos^2 x - \sin x \cos x + \tan x}{\cos^2 x + \sin x \cos x - \tan x} &= \frac{\cos^2 x - \sin x \cos x + \tan x}{\cos^2 x + \sin x \cos x - \tan x} \cdot \frac{\cos x}{\cos x}\\ &= \frac{\cos^3 x - \sin x \cos^2 x + \sin x}{\cos^3 x + \sin x \cos^2 x - \sin x}\\ &= \frac{\cos^3 x + (\sin x)(1 - \cos^2 x)}{\cos^3 x - (\sin x)(1 - \cos^2 x)}\\ &= \frac{\cos^3 x + \sin^3 x}{\cos^3 x - \sin^3 x} = \frac{1 + \tan^3 x}{1 - \tan^3 x}.\end{aligned}$$

(c) Whenever x is not an integer multiple of $\frac{\pi}{2}$, we have

$$\frac{\sin 3x}{\sin x}+\frac{\cos 3x}{\cos x}=\frac{\sin 3x\cos x+\cos 3x\sin x}{\sin x\cos x}=\frac{\sin(3x+x)}{\frac{1}{2}\sin 2x}$$
$$=\frac{\sin 4x}{\frac{1}{2}\sin 2x}=\frac{2\sin 2x\cos 2x}{\frac{1}{2}\sin 2x}=4\cos 2x.$$

(d) Whenever x is not an odd integer multiple of $\frac{\pi}{2}$, we have

$$1-\sec^2 x+\sec x\tan x=1-\frac{1}{\cos^2 x}+\frac{1}{\cos x}\frac{\sin x}{\cos x}=\frac{\cos^2 x-1+\sin x}{\cos^2 x}=\frac{-\sin^2 x+\sin x}{\cos^2 x}$$
$$=\frac{\sin x-\sin^2 x}{1-\sin^2 x}=\frac{\sin x\,(1-\sin x)}{(1+\sin x)(1-\sin x)}=\frac{\sin x}{1+\sin x}.$$

18. If $x\in(-\pi,\pi)$ and if $t=\tan\frac{x}{2}$, then

$$\frac{1-t^2}{1+t^2}=\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}=\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}=\frac{\cos\left(2\cdot\frac{x}{2}\right)}{1}=\cos x$$

and

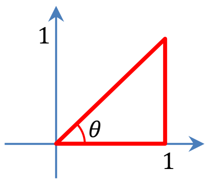
$$\frac{2t}{1+t^2}=\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}=\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}=\frac{\sin\left(2\cdot\frac{x}{2}\right)}{1}=\sin x.$$

19. (a) Note that $\sin\frac{5\pi}{3}=\sin\left(\frac{5\pi}{2}-\frac{5\pi}{6}\right)=\cos\frac{5\pi}{6}$ and the number $\frac{5\pi}{6}\in[0,\pi]$. Thus

$$\arccos\left(\sin\frac{5\pi}{3}\right)=\arccos\left[\sin\left(\frac{5\pi}{2}-\frac{5\pi}{6}\right)\right]=\arccos\left(\cos\frac{5\pi}{6}\right)=\frac{5\pi}{6}.$$

(b)

$$\sec(\arctan 1)=\sec\left(\frac{\pi}{4}\right)=\frac{1}{\cos\frac{\pi}{4}}=\sqrt{2}.$$



(c) Note that $\cos 10=\cos(4\pi-10)$ and the number $4\pi-10\in[0,\pi]$. Thus

$$\arccos(\cos 10)=\arccos(\cos(4\pi-10))=4\pi-10.$$

20. (a) $\sinh$: Domain $= \mathbb{R}$, range $= \mathbb{R}$

$\cosh$: Domain $= \mathbb{R}$, range $= [1, +\infty)$

$\tanh$: Domain $= \mathbb{R}$, range $= (-1, 1)$

(b) For every $x \in \mathbb{R}$ we have

$$\sinh(-x) = \frac{e^{-x} - e^{-(-x)}}{2} = -\frac{e^{x} - e^{-x}}{2} = -\sinh x\,,$$

so $\sinh$ is an odd function. $\sinh$ is not an even function.

For every $x \in \mathbb{R}$ we have

$$\cosh(-x) = \frac{e^{-x} + e^{-(-x)}}{2} = \frac{e^{x} + e^{-x}}{2} = \cosh x\,,$$

so $\cosh$ is an even function. $\cosh$ is not an odd function.

For every $x \in \mathbb{R}$ we have

$$\tanh(-x) = \frac{e^{-x} - e^{-(-x)}}{e^{-x} + e^{-(-x)}} = -\frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} = -\tanh x\,,$$

so $\tanh$ is an odd function. $\tanh$ is not an even function.